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Sequential Estimation of the State and the Observation-Error Covariance Matrix

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The simultaneous maximum-likelihood estimate of the observation-error covariance matrix and the state is developed in sequential form. The resulting estimator requires only an approximate initial estimate for the observation-error covariance matrix, and the estimator uses the observation data to refine and update the initial estimate. Simulated observation data of a near-Earth satellite are used to evaluate the estimation scheme. The observation vector consists of a simultaneous measurement of range, azimuth, and elevation for a spacecraft during one pass over a tracking station. The results of the numerical simulation indicate that if the a priori value for the observation-error covariance matrix is in error, the performance of the estimation algorithm is significantly improved when the covariance matrix is estimated.

Nomenclature

$A(t)$ = $n \times n$ matrix of time-dependent coefficients
C = constant
cof = cofactor
F = log of the likelihood function
$f(x,y;R)$ = joint density function
H = mapping matrix
I = identity matrix
i,j,k,l,N,p = indices
$L(x,R)$ = likelihood function
P = estimation-error covariance matrix
R = observation-error covariance matrix
$\hat{R}_0$ = initial guess of R
R_T = true value of R
t = time
v = observation-noise vector
W = weighting function
X = true state
X^* = reference state
x = state deviation
Y = actual observation
y = observation deviation
$y_1,\ldots,y_N$ = set of observation deviations
σ = standard deviation
σ^2 = variance
ϕ = null matrix
$\Phi(t_k,t_j)$ = state transition matrix
$(\hat{\ })$ = best estimate
$(\ )^*$ = evaluated on the reference trajectory

Subscript

0 = initial value

Superscript

T = transpose

Operators

E = expected value
$(\dot{\ })$ = differentiation with respect to time

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Introduction

THE problem of estimating the state of a spacecraft whose initial state is unknown and whose motion is influenced by inaccurately modeled or unknown forces is referred to as an orbit determination problem. An orbit determination problem is solved by using observations of the motion of the spacecraft. The true values of the observations are usually nonlinear functions of the state. In practice, the data obtained by tracking or observing the space vehicle are corrupted by random observation errors. In the usual solution to an orbit determination problem, a priori statistics on the errors in the initial state of the spacecraft and on the errors in the observations are assumed to be available. The observation error is assumed to be distributed with zero mean and a known covariance. However, for various reasons, the true error covariance may differ significantly from the a priori covariance. Using the incorrect observation-error covariance may lead to an estimate of the state which differs significantly from the true state.

In Ref. 1, a formulation of the estimation algorithm is given in which an estimate of the variance of the observation error is made simultaneously with the estimate of the state. The solution to the orbit determination problem is restricted to a single scalar observation, and it is obtained by assuming that the variance of the observation error is a random variable whose distribution is characterized by an inverted gamma distribution function.

In many orbit determination problems, simultaneous measurements, such as range and range rate, are made at each observation time. The set of observations taken at each point in time may be correlated. In this paper, an estimation algorithm is given which allows a simultaneous estimate of the state of the spacecraft and the covariance matrix associated with a general observation-error vector. The solution to the orbit determination problem is given in the form of a "batch estimator," and an alternate method of formulation of the estimation algorithm is given which allows sequential processing of the tracking data. The resulting sequential estimation algorithm is applied to the problem of estimating the orbit of a near-Earth satellite using simulated observation data.

Estimation Algorithm

If the n vector x_k is defined as the difference between the true state X_k and a reference state X_k^* and if x_k is sufficiently small, then x_k is related to any other x_j by the relation

$$x_k = \Phi(t_k,t_j)x_j \tag{1}$$

The $n \times n$ matrix $\Phi(t_k,t_j)$ is referred to as the state transition matrix and satisfies the matrix differential equation

$$\dot{\Phi}(t,t_j) = A(t)\Phi(t,t_j), \quad \Phi(t_j,t_j) = I \tag{2}$$

where I is the $n \times n$ identity matrix. If the reference trajectory is defined by the conditions $\dot{X} = F(X,t)$, $X(t_0) = X_0^*$, the $n \times n$ matrix of time-dependent coefficients $A(t)$ is defined as $A(t) = (\partial F/\partial X)^*$. The superscript $(\ )^*$ indicates that the matrix of partial derivatives is evaluated on the reference trajectory.

It is assumed that $y_1,\ldots,y_N$ is a set of observation deviations where the p vector $y_i = Y_i - Y_i^*$; that is, y_i is the difference between the observation on the reference trajectory and the actual observation. If x_i is sufficiently small and if $Y_i = G(X_i,t_i)$, the relationship between y_i and x_i is approximated by the linear expression

$$y_i = \tilde{H}_i x_i + v_i, \quad i = 1,\ldots,N \tag{3}$$

where $\tilde{H}_i = (\partial G/\partial X_i)^*$ and where the p vector v_i is a member of a population whose distribution is $N(\phi,R)$; that is, $E[v_i] = \phi$ and $E[v_i v_i^T] = R$. The matrix R is assumed to be unknown, but constant, during the sample period, $t_1 \leq t \leq t_N$. If H_i is defined as

$$H_i = \tilde{H}_i\Phi(t_i,t_k) \tag{4}$$

Equation (3) may be expressed as

$$y_i = H_i x_k + v_i, \quad i = 1,\ldots,N \tag{5}$$

where x_k is the deviation of the state from the nominal trajectory at a particular reference time t_k.

In the usual orbit determination problem x_k is unknown because the initial state of the spacecraft is unknown, and because errors exist in the mathematical model used to describe the motion of the spacecraft. The actual injection conditions x_0 are assumed to be a random vector whose a priori statistics are $N(\phi,P_0)$.

The problem considered in this paper is stated as follows: given the sequence of observations $y_1,\ldots,y_N$, the relation $y_i = H_i x_k + v_i$ where v_i has a distribution of $N(\phi,R)$ with R an unknown constant matrix, and the a priori statistics on the injection conditions, that is, $E[x_0] = \phi$ and $E[x_0x_0^T] = P_0$, find the best estimate of x_k and R. The best estimate of x_k and of R, denoted as $\hat{x}_k$ and $\hat{R}$, respectively, is defined to be the maximum-likelihood estimate of x_k and R.

If $F(x,R) = \log L$ where $L(x,R)$ is the likelihood function of x and R, then for the assumed statistics on v_i and x_0, $F(x,R)$ is given as

$$F(x_k,R) = C + \frac{N}{2}\log|R^{-1}| - \tfrac{1}{2}\sum_{i=1}^{N}(y_i - H_i x_k)^T R^{-1} \times (y_i - H_i x_k) + (x_k - \bar{x}_k)^T P_{k/0}^{-1}(x_k - \bar{x}_k) \tag{6}$$

where C is a constant, $\bar{x}_k = \Phi(t_k,t_0)\bar{x}_0$, and $P_{k/0} = \Phi(t_k,t_0) P_0\Phi^T(t_k,t_0)$. The maximum-likelihood estimate requires that $F(x_k,R)$ be maximized with respect to the choice of x_k and R. To obtain the maximum of $F(x_k,R)$, the conditions

$$\partial F/\partial x_i = 0, \quad i = 1,\ldots,n \tag{7}$$

and

$$\partial F/\partial R_{ij} = 0, \quad i,j = 1,\ldots,p \tag{8}$$

must be satisfied. From Eq. (7), $\hat{x}_k$ is obtained in a straightforward manner as

$$\hat{x}_k = \left(\sum_{i=1}^{N} H_i^T\hat{R}^{-1}H_i + P_{k/0}^{-1}\right)^{-1}\sum_{i=1}^{N} H_i^T\hat{R}^{-1}y_i \tag{9}$$

where the condition $\bar{x}_k = \phi$ has been used and where the matrix that is to be inverted is assumed to be full rank. Following a procedure similar to that used to obtain Lemma 3.2.2 of Ref. 2, the set of conditions represented in Eq. (8) is used to obtain the result

$$\hat{R} = \frac{1}{N}\sum_{i=1}^{N}(y_i - H_i\hat{x}_k)(y_i - H_i\hat{x}_k)^T \tag{10}$$

The derivation of Eqs. (9) and (10) is discussed in the Appendix. The estimation algorithm represented by Eqs. (9) and (10) is in the form of a "batch processor," in which the entire data set is processed to obtain an estimate of the state and of the covariance at a particular reference time t_k.

Since Eqs. (9) and (10) depend on both $\hat{x}_k$ and $\hat{R}$ in a nonlinear manner, the solution must be obtained iteratively. That is for each reference trajectory the iteration is carried out to obtain an $\hat{x}_k$ and $\hat{R}$ which satisfies Eqs. (9) and (10). The iteration requires assuming an initial value for $\hat{R}$ and then computing $\hat{x}_k$. Then, $\hat{x}_k$ is used to compute a new value for $\hat{R}$. The process is repeated until the values for $\hat{x}_k$ and $\hat{R}$ no longer change. Then a new reference trajectory is selected to correct for the estimated error $\hat{x}_k$ and the entire process repeated until convergence has been obtained. A converged estimate is one for which $|\hat{x}_k{}^i - \hat{x}_k{}^{i-1}| \leq \epsilon$, where ϵ is a small number.

Since Eqs. (9) and (10) are both nonlinear functions of $\hat{x}_k$ and $\hat{R}$, the property of unbiasedness cannot be established. However, the asymptotic properties of the estimators for x_k and R under the assumption that x and R are parameters are discussed in Ref. 4. The asymptotic properties of the estimators assuming that a priori information on x and R is available and that R is a diagonal matrix are discussed in Ref. 5. In both references, the estimators are shown to be asymptotically unbiased. However, in most orbit determination situations the data set is finite, and the characteristics of the estimation algorithm for a finite data set must be determined through simulated numerical studies.

Recursive Estimation Algorithm

The relations for the batch estimator given by Eqs. (9) and (10) can be expressed in recursive form. If the subscript k is replaced by N and the definition

$$P_N = \left(\sum_{i=1}^{N} H_i{}^T\hat{R}_i{}^{-1}H_i + P_{N/0}{}^{-1}\right)^{-1} \tag{11}$$

is used, then $P_N{}^{-1}$ can be expressed as follows

$$P_N{}^{-1} = H_N{}^T\hat{R}_N{}^{-1}H_N + \sum_{i=1}^{N-1} H_i{}^T\hat{R}_i{}^{-1}H_i + P_{N/0}{}^{-1} = H_N{}^T\hat{R}_N{}^{-1}H_N + \bar{P}_N{}^{-1} \tag{12}$$

where

$$\bar{P}_N{}^{-1} = \sum_{i=1}^{N-1} H_i{}^T\hat{R}_i{}^{-1}H_i + P_{N/0}{}^{-1}$$

Hence, from Eq. (12) it follows that

$$P_N = (\bar{P}_N{}^{-1} + H_N{}^T\hat{R}_N{}^{-1}H_N)^{-1} \tag{13}$$

By using the matrix identity given in Ref. 3, Eq. (13) can be expressed as

$$P_N = (I - K_NH_N)\bar{P}_N \tag{14}$$

where

$$K_N = \bar{P}_NH_N{}^T(H_N\bar{P}_NH_N{}^T + \hat{R}_N)^{-1}$$

After algebraic rearrangement, Eq. (9) is expressed as

$$\hat{x}_N = P_NH_N{}^T\hat{R}_N{}^{-1}y_N + P_N\bar{P}_N{}^{-1}\bar{x}_N \tag{15}$$

where

$$\bar{x}_N = \Phi(t_N,t_{N-1})\hat{x}_{N-1} = \bar{P}_N \sum_{i=1}^{N-1} H_i\hat{R}_i{}^{-1}y_i$$

that is $\bar{x}_N$ is the estimate of x_N based on $(N - 1)$ observations. Substituting Eq. (14) for P_N in Eq. (15) leads to the result

$$\hat{x}_N = \bar{x}_N + K_N(y_N - H_N\bar{x}_N) \tag{16}$$

The primary difference between Eq. (16) and the results given in Ref. 3 is that the covariance R is replaced by $\hat{R}_N$, the estimate of R at the Nth observation point.

To obtain a sequential estimation algorithm for R, Eq. (10) is expressed as

$$\hat{R}_N = \frac{1}{N}(y_N - H_N\hat{x}_N)(y_N - H_N\hat{x}_N)^T + \frac{1}{N}\frac{(N-1)}{(N-1)}\sum_{i=1}^{N-1}(y_i - \tilde{H}_i\hat{x}_{i/N})(y_i - \tilde{H}_i\hat{x}_{i/N})^T \tag{17}$$

where

$$H_i\hat{x}_N = \tilde{H}_i\Phi(t_i,t_N)\hat{x}_N = \tilde{H}_i\hat{x}_{i/N}$$

If the second term of the right-hand expression of Eq. (17) is approximated by the estimate of R for $N - 1$ data points [that is, $\hat{R}_{N-1}$ multiplied by the factor $(N - 1)/N$)], Eq. (17) can be expressed in sequential form as

$$\hat{R}_N = (1 - 1/N)\hat{R}_{N-1} + (1/N)\bar{v}_N\bar{v}_N{}^T \tag{18}$$

where $\bar{v}_N = y_N - H_N\bar{x}_N$. Equations (16) and (18) are combined with Eq. (14), and the usual relations for propagating the estimate and the covariance that is, $\bar{x}_k = \Phi(t_k,t_{k-1})\hat{x}_{k-1}$, and

$$\bar{P}_k = \Phi(t_k,t_{k-1})P_{k-1}\Phi^T(t_k,t_{k-1})$$

to obtain the sequential estimation algorithm. The advantage of using Eq. (18) rather than Eq. (10) is that only the last data point need be processed at each observation time, while Eq. (10) requires processing the entire data sequence for each value of N. Equation (18) is an approximation for Eq. (17) because $\hat{R}_{N-1}$ involves only the observations up through y_{N-1}. However, the $\hat{x}_{i/N}$ which appears in Eq. (17) is the estimate of x_i based on N observations rather than $N - 1$ observations. The approximation, then, involves replacing the estimate of $\hat{x}_{i/N}$ with $\hat{x}_{i/N-1}$. For large N, $\hat{x}_{i/N}$, and $\hat{x}_{i/N}$ will differ by an insignificant amount. It should be noted, however, that this approximation is made for each data point, and the total error will be the cumulative effect of the approximation made during the processing of each data point. For the numerical example considered in this investigation, convergence to the correct R with the sequential form of the estimation algorithm occurred within four iterations of the data. This iteration process is described in the results section.

When N is small, the estimate for R is greatly influenced by the initial estimate of the state. Since, in this case, the estimate of the state may not be good, large variations in the estimate of R may result. Hence, modification of the weighting function $1/N$ is desirable. The modification should allow an opportunity for the sequential estimate of the state to stabilize before the sequential estimate is used to correct the initial estimate of R. An acceptable modification to the algorithm for estimating R can be obtained by expressing Eq. (18) as

$$\hat{R}_N = (1 - W_N)\hat{R}_{N-1} + W_N\bar{v}_N\bar{v}_N{}^T \tag{19}$$

where

$$W_N = (N - 1)(N - 2)(N - 3)\ldots(N - k)/N^{k+1} \tag{20}$$

where k is an integer. The weighting function W is zero for all $N \leq k$, and as $N \to \infty$, W approaches $(1/N)$. Hence for $N \leq k$, the value of R is not changed from the a priori value. The choice for k will depend upon how well $\hat{R}_0$ is known, i.e., the more accurate $\hat{R}_0$, the larger k may be.

Computational Algorithm

The computational algorithm for the sequential estimation algorithm can be summarized as follows: 1) choose P_0, $\hat{x}_0$, $\hat{R}_0$ and k; 2) set $N = 1$; 3) compute $\bar{x}_N$ from $\bar{x}_N = \Phi(t_N,t_{N-1})\hat{x}_{N-1}$; 4) compute W_N and $\hat{R}_N$ from a) $W_N = (N - 1)(N - 2)\ldots(N - k)/N^{k+1}$ and b) $\hat{R}_N = (1 - W_N)\hat{R}_{N-1} + W_N(y_N - H_N\bar{x}_N)(y_N - H_N\bar{x}_N)^T$; 5) compute $\hat{x}_N$ from a) $\bar{P}_N = \Phi(t_N,t_{N-1})P_{N-1}\Phi^T(t_N,t_{N-1})$, b) $K_N = \bar{P}_NH_N{}^T(H_N\bar{P}_NH_N{}^T + \hat{R}_N)^{-1}$, and c) $\hat{x}_N = \bar{x}_N + K_N(y_N - H_N\bar{x}_N)$; 6) return to step 4b and replace $\bar{x}_N$ with $\hat{x}_N$; 7) repeat steps 5b and 5c; 8) repeat steps 6 and 7 until there is no change in $\hat{R}_N$ and $\hat{x}_N$; and 9) set $N = N + 1$ and return to step 3.

Note that the only iteration involved in the sequential processor is steps 6 and 7. However, for the example considered in this paper, $\hat{x}_N$ and $\hat{R}_N$ did not change after the third iteration. In fact, steps 6 and 7 could have been eliminated and only $\bar{x}_N$ used at each point to obtain satisfactory results.

After the entire data set has been processed a new nominal can be obtained by replacing $X_0{}^*$ with $X_0{}^* + \hat{x}_0$, and the entire data set reprocessed to correct for the effects of nonlinearities in the state-propagation and observation-state relations.

Results

The recursive estimation algorithm described in the previous section was used to estimate the state vector and the observation-error covariance matrix for the noise in a radar system tracking a near-Earth satellite. Observations of range, azimuth, and elevation were simulated at 6-sec intervals for a satellite tracked by radar at the Bermuda tracking station. Only those observations corresponding to elevations above 5° were processed. The resulting data set consisted of 42 observation vectors, each composed of one observation of range, azimuth, and elevation. The simulated observations were corrupted by the addition of Gaussian noise that had a mean of zero and variances of

$$\sigma_R^2 = 400 \text{ ft}^2, \sigma_A^2 = \sigma_E^2 = 0.625 \times 10^{-3} \text{ deg}^2 \quad (21)$$

All covariances for the observation noise were zero.

The initial conditions, corresponding to a typical Gemini trajectory, which were used to generate the simulation data were

$$x_0 = 0.15339388 \times 10^7 \text{ ft}, \quad y_0 = -0.18477064 \times 10^8 \text{ ft}$$
$$z_0 = 0.10753242 \times 10^8 \text{ ft}, \quad \dot{x}_0 = 0.24757653 \times 10^5 \text{ fps} \quad (22)$$
$$\dot{y}_0 = 0.49646847 \times 10^4 \text{ fps}, \quad \dot{z}_0 = 0.49928933 \times 10^4 \text{ fps}$$

The mean orbital altitude is 110 miles. The initial conditions used for the nominal trajectory were 1000 ft greater than each of the previously mentioned position coordinates and 5 fps greater than each velocity component. The orbit was assumed to be perturbed by the zonal harmonics J_{20}, J_{30}, and J_{40}. The effects of atmospheric drag and lunar and solar perturbations were neglected.

To initiate the estimation procedure, an initial guess for $\hat{x}_0$ and P_0 is required. For the problem presented in this paper, the initial guess for $\hat{x}_0$ was assumed to be the null vector, and the initial variances of the position estimation errors were assumed to be 0.4×10^6 ft^2, while the initial velocity variances were chosen to be 30 ft^2/sec^2. The initial covariances were chosen to be zero.

Also, an initial guess $\hat{R}_0$ is required for R. If a value of $\hat{R}_0$ is chosen which is much smaller than R_T, the true value of R, the value of P will decrease rapidly. As the value of P approaches zero, the estimator neglects subsequent observation data. However, if the subsequent observation data are ignored, large errors in the estimate of the state may result. To avoid this problem, $\hat{R}_0$ should be chosen large enough to insure that its value is greater than the value of R_T; the true but unknown value of R. In this example R_T corresponds to the value of R used to generate the observation data. Several values of $\hat{R}_0$ were tried, and it was found that the estimate of the state was relatively insensitive to the value for $\hat{R}_0$ in the range $R_T \leq \hat{R}_0 \leq 4 \times R_T$. Consequently, the results shown here are for $\hat{R}_0 = 15 \times R_T$.

The errors in the standard deviation of range, azimuth, and elevation for five iterations of the data are presented in Fig. 1. The error is defined as the difference between the true value of the standard deviation and that obtained from the estimation algorithm. For example, the error in standard deviation associated with the initial value of range is $20 - (15 \times 20) = -280$ ft^2. As indicated by Fig. 1, the process converged after four iterations, and the fifth iteration essentially duplicates the fourth.

This iteration is accomplished by processing all data and then returning to step 1 of the computational algorithm. The observation data is then reprocessed using the last value of $\hat{R}_N$ from the previous iteration as $\hat{R}_0$. After each iteration the initial conditions for the nominal trajectory were updated to improve the linearity assumptions. The values of P_0 were identical for each iteration. This iteration of the entire

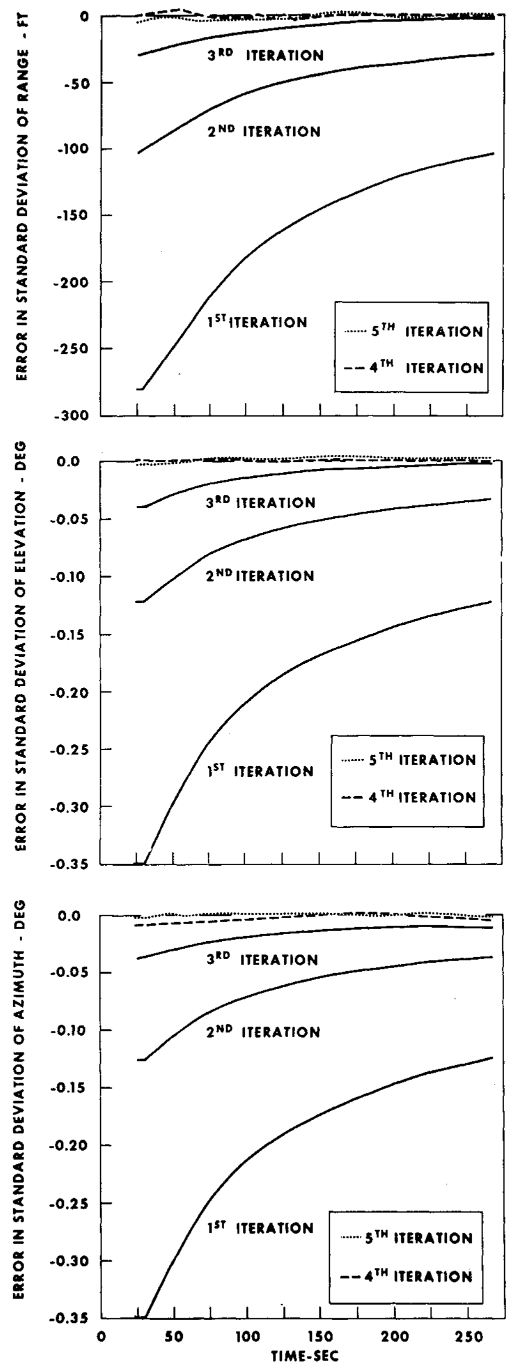

Fig. 1 Error in standard deviation of range, elevation, and azimuth for five iterations.

data set should not be confused with the iteration of each data point associated with steps 6 and 7 of the computational algorithm.

It is emphasized that iteration of the data set would be impractical in a real time tracking situation but could be used for post flight analysis. However, the method should be more rapidly convergent than the batch processor which obtains an improved estimate of R only after processing the entire data set.

Plots showing the position and velocity error metrics of the first and fifth iterations of the data are shown in Figs. 2 and 3. The position-error metric is defined as

$$\Delta r = (\Delta x^2 + \Delta y^2 + \Delta z^2)^{1/2} \quad (23)$$

where Δx, Δy, and Δz are the differences between the true and estimated values of x, y, and z. Recall that x, y, and z are defined as the differences between the geocentric inertial com-

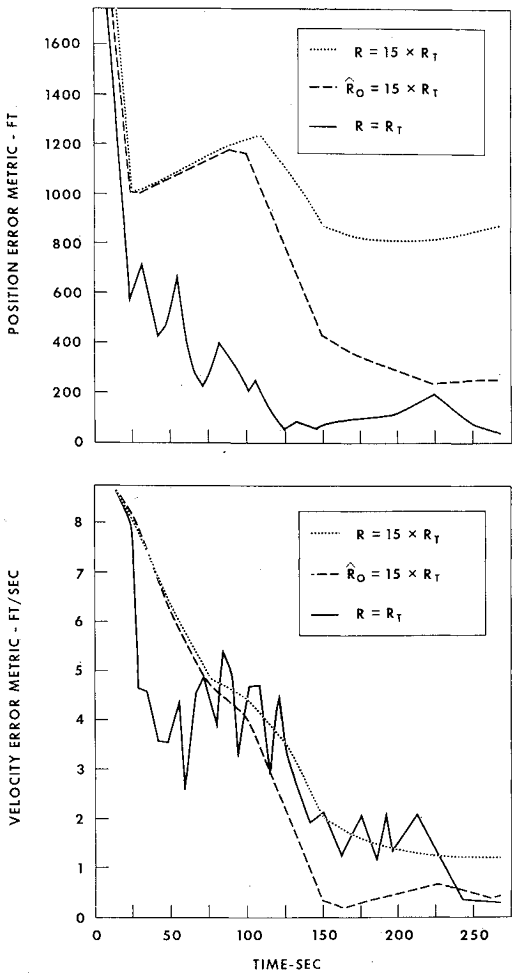

Fig. 2 Comparison of position and velocity error metrics for true, incorrect, and sequentially estimated observation-error covariance matrix—first iteration.

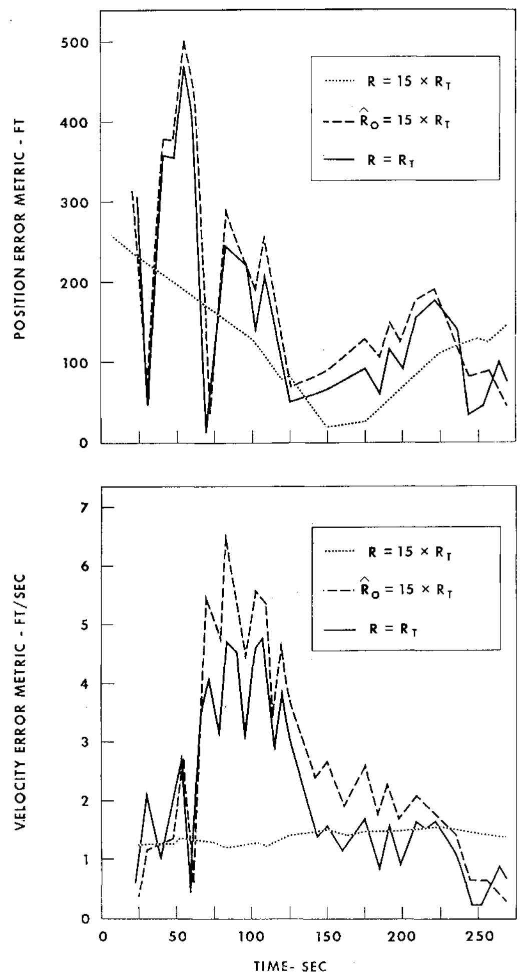

Fig. 3 Comparison of position and velocity error metrics for true, incorrect, and sequentially estimated observation-error covariance matrix—fifth iteration.

ponents X, Y and Z of the true and nominal trajectories, respectively. A similar definition applies for the velocity metric with the position components replaced by velocity components in Eq. (23). The results in Figs. 2 and 3 are shown for $R = R_T$, $R = 15 \times R_T$, and for R sequentially estimated by Eq. (19) with $\hat{R}_0 = 15 \times R_T$ and $k = 2$. An examination of the equations for the sequential estimator of the state reveals that small observation error variances cause the estimator to place more emphasis on the observation data than the estimator places on the previous best estimate of the state. The estimator will continue to place more emphasis on the observation data until the filter begins to saturate, i.e., the gain matrix K approaches zero. Conversely, if R is large, the filter will be influenced more by the previous best estimate of the state. The results shown in Figs. 2 and 3 display this characteristic since the error metrics for the condition where $R = R_T$ are much more sensitive to individual observations than the condition where $R = 15 \times R_T$.

The question of iteration at each data point associated with steps 6 and 7 of the computational procedure is illustrated by the results shown in Table I. Table I presents the values of $\hat{x}$ during four iterations for one data point. Examination of the values in the table indicates that the iteration process converged after three iterations. It was noted that convergence within three iterations was obtained for all data points considered. In fact, the value of $\bar{x}_N$ could have been used at each point, and the pointwise iteration process omitted to obtain acceptable numerical results.

Finally, as previously discussed, the choice for k in Eq. (20) is dependent upon how well $\hat{R}_0$ is known, i.e., the more accurate R_0 the larger k may be. It was found that a value for k of two or three worked very well for values of $\hat{R}_0$ in the range of $R_T \leq \hat{R}_0 \leq 15 \times R_T$ for this example.

Conclusions

Algorithms for estimating the state and observation-error covariance matrix have been developed in batch and sequential processor form. For the problem considered in this paper —estimating the orbit of a near-Earth satellite by using simulated observation data—the iterative process for the sequential algorithm was found to be rapidly convergent. The results of the numerical studies conducted in this investigation indicate that the observational data can be used to improve the estimate of the variance in the observation noise, and that the improved estimate of the observation noise variance will lead to an improved estimate of the state.

Appendix

The best estimates of x_k and R are defined as the values which maximize the log of the joint density function $f(x,y,;R)$. If $F(x,R) = \log f(x,y;R)$

$$F(x,R) = -\tfrac{1}{2} Np \log 2\pi + \frac{N}{2} \log |W| - \tfrac{1}{2} \sum_{i=1}^{N} (y_i - H_i x_k)^T W (y_i - H_i x_k) + (x_k - \bar{x}_k)^T P_k^{-1} (x_k - \bar{x}_k) \quad \text{(A1)}$$

Table 1 Typical values for $\hat{x}$ for one point during four iterations

Iteration	x, ft	y, ft	z, ft	$\dot{x}$, fps	$\dot{y}$, fps	$\dot{z}$, fps
1	1471.5663	1022.3438	184.6865	4.3650742	3.0891238	0.48101786
2	1471.5821	1022.6041	184.6649	4.3673150	3.0913560	0.48118874
3[a]	1471.5822	1022.6053	184.6649	4.3673256	3.0913665	0.48118961
4	1471.5822	1022.6053	184.6649	4.3673256	3.0913665	0.48118961

[a] Process has converged to eight places after three iterations.

where the identity $W = R^{-1}$ has been used. If R is nonsingular, a unique determination of W will determine R. Necessary conditions for a minimum of Eq. (A1) with respect to the choice of x_k and W are

$$\partial F/\partial x_{ik} = 0 \qquad i = 1,\ldots,n \tag{A2}$$

and

$$\partial F/\partial W_{jl} = 0 \qquad j,l = 1,\ldots,p \tag{A3}$$

Eq. (A2) is expressed in vector form as $(\partial F/\partial x_k) = \phi$, and this condition leads to the expression

$$\left(\frac{\partial F}{\partial x_k}\right) = \phi = -\sum_{i=1}^{N} (y_i - H_i\hat{x}_k)^T W H_i + (\hat{x}_k - \bar{x}_k)^T \bar{P}_k^{-1} \tag{A4}$$

Rearranging the transpose of Eq. (A4) leads to

$$\left(\sum_{i=1}^{N} H_i^T W H_i + \bar{P}_k^{-1}\right)\hat{x}_k = \sum_{i=1}^{N} H_i^T W y_i + \bar{P}_k^{-1}\bar{x}_k \tag{A5}$$

Recalling that $\bar{x}_k = \Phi(t_k,t_0)\hat{x}_0 = \phi$ if $\hat{x}_0 = \phi$, and substituting $W = R^{-1}$ and assuming that the $n \times n$ coefficient matrix for $\hat{x}_k$ is nonsingular allows expression of $\hat{x}_k$ as

$$\hat{x}_k = \left(\sum_{i=1}^{N} H_i^T \hat{R}^{-1} H_i + \bar{P}_k^{-1}\right)^{-1} \sum_{i=1}^{N} H_i^T \hat{R}^{-1} y_i \tag{A6}$$

From eq. (A3) $(\partial F/\partial W_{jl} = 0)$, the condition

$$\frac{\partial F}{\partial W_{jl}} = 0 = 2\left[\frac{N}{2}\frac{1}{|W|}\frac{\partial |W|}{\partial W_{jl}} - \frac{1}{2}\sum_{i=1}^{N} (y_i^j - H_i^j\hat{x}_k)(y_i^l - H_i^l\hat{x}_k)\right] \tag{A7}$$

is obtained where y_i^j indicates the jth component of the vector y_i, and H_i^j indicates the jth row of the matrix H_i. The factor 2 occurs because $W_{jl} = W_{lj}$. Noting that

$$|W| = \sum_{j=1}^{p} W_{jl}\,\mathrm{cof}(W_{jl})$$

where $\mathrm{cof}(W_{jl})$ indicates the cofactor of W_{jl}, the condition

$$\partial |W|/\partial W_{jl} = \mathrm{cof} W_{jl} \tag{A8}$$

follows. Then $\partial F/\partial W_{jl} = 0$ leads to the condition

$$N\,\frac{\mathrm{cof}(W_{jl})}{|W|} = \sum_{i=1}^{N} (y_i^j - H_i^j\hat{x}_k)(y_i^l - H_i^l\hat{x}_k) \tag{A9}$$

In a similar manner, if $j = l$, $\partial F/\partial W_{ll} = 0$ leads to

$$\frac{N}{2}\frac{\mathrm{cof}(W_{ll})}{|W|} = \tfrac{1}{2}\sum_{i=1}^{N} (y_i^l - H_i^l\hat{x}_k)^2 \tag{A10}$$

Noting that $\mathrm{cof}(W_{jl})/|W|$ is the jlth element of W^{-1}, Eqs. (A9) and (A10) are combined to obtain

$$\hat{W}^{-1} = \frac{1}{N}\sum_{i=1}^{N} (y_i - H_i\hat{x}_k)(y_i - H_i\hat{x}_k)^T \tag{A11}$$

The combination of Eqs. (A6) and (A11) with the condition that $W^{-1} = R$, yields Eqs. (9) and (10).

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